ON MAPS OF FINITE COMPLEXES INTO NILPOTENT SPACES OF FINITE TYPE: A CORRECTION TO 'HOMOTOPICAL LOCALIZATION'

By PETER HILTON, GUIDO MISLIN, *and* JOSEPH ROITBERG

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1. Introduction

In March 1976, one of the authors (P. H.) received a letter from Frank Adams in which Adams cast serious doubt on the validity of Corollary 2.2(c) of [**2**], while accepting Theorem 2 of [**1**]. In fact, Theorem 2 of [**1**] was reproduced as Corollary 3.5 of [**2**]; and Corollaries 2.2(c) and 3.5 of [**2**] reappeared, in identical form, in the monograph [**3**], written by the same authors, as Corollary II.5.4(c) and Corollary II.5.11. Let us reproduce those two statements as they appeared in [**2, 3**]; we will adopt the numbering of [**3**].

STATEMENT 1 (Corollary II.5.4(c) of [**3**]). *Suppose W is a connected finite CW-complex and X is a nilpotent CW-complex of finite type. Then there exists a cofinite set of primes Q such that the canonical map $[W, X_Q] \to [W, X_0]$ is one-to-one.*

STATEMENT 2 (Corollary II.5.11 of [**3**]). *Suppose W is a connected finite CW-complex and X is a nilpotent CW-complex of finite type. Given a map† $f\colon W \to X_0$, there exists a cofinite set of primes Q such that f factors uniquely as $f = r_Q g$, where $g\colon W \to X_Q$ and $r_Q\colon X_Q \to X_0$ is the canonical map.*

We repeat that Adams accepted Statement 2 but not Statement 1; indeed, he provided, in his letter, a proposed counter-example to Statement 1, assuming certain detailed statements could be checked. Adams was, of course, right. Statement 1 is incorrect, while Statement 2 is correct; and Adams' proposed counter-example is, indeed, a counter-example. However, the situation was complicated for us by the fact that we based our proof of Statement 2 on Statement 1 (together with Corollary II.5.10 of [**3**], identical with Corollary 3.4 of [**2**] and correct!).

† It was clear that all statements were made 'up to homotopy'.

Indeed, Statement 2 was the *only* consequence we drew from Statement 1 in both [**2**] and [**3**]; and we did not offer a detailed proof of Statement 1 at all, believing it followed automatically from a line of reasoning we had employed in obtaining the principal result of [**2**].

We have found the error most instructive. For, as we point out in § 2, Statement 1 becomes correct if we impose some homogeneity on the non-empty counter-images of $[W, X_Q] \to [W, X_0]$, as would be obtained if W were a suspension or X a rational H-space. Statement 1 also becomes correct if we simply ask that $[W, X_Q] \to [W, X_0]$ be *weakly injective*, meaning that the counter-image of the class of the constant map should consist only of the class of the constant map. Its failure in general derives from the fact that there may be so much inhomogeneity in the counter-images; we may give some precision to this idea as follows. Suppose that $S \subseteq T$ denote sets of primes, suppose that $W = V \cup e^n$ for $n \geqslant 2$, let $g\colon W \to X_T$, and let $\bar{g} = g \mid V$. We then have a commutative diagram (see (2.1) or the proof of Theorem II.5.3 of [**3**])

$$(1.1)\qquad \begin{array}{ccccc} \pi_1(X_T^V, \bar{g}) & \xrightarrow{\varphi_g} & \pi_n X_T & \longrightarrow & [W, X_T]_g \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e_*} \\ \pi_1(X_S^V, e\bar{g}) & \xrightarrow{\psi_{eg}} & \pi_n X_S & \longrightarrow & [W, X_S]_{eg} \end{array}$$

where we have written e_* for the map induced by $e\colon X_T \to X_S$, and we have written e, generically, for a localizing map; and the suffixes g, eg indicate our chosen base points for the given homotopy sets. Then, as Adams' counter-example shows, the order of the kernel of the localizing map $e\colon \operatorname{coker}\varphi_g \to \operatorname{coker}\psi_{eg}$ may depend on g and, indeed, the primes appearing in that order may run over the entire set of primes as g varies (when T is the entire set of primes). Of course, $\operatorname{coker}\varphi_g$ is a subset of $[W, X_T]_g$, mapped to $\operatorname{coker}\psi_{eg}$ by e_*, so that the possibility of gross inhomogeneity is established!

As the numbering suggests, Statement 1 is the third part of a three-part statement, Corollary II.5.4(a), (b), (c) of [**3**]. Corollary II.5.4(b) is essentially equivalent to Corollary II.5.4(c) and fails with it; we will pay no further attention to it. In § 2 we detail the proof of Corollary II.5.4(a), since no details were given in [**2**] or [**3**] and we prove the weakened forms of Statement 1. We also give a detailed proof of Statement 2; in fact, to do so we actually strengthen Statement 2 slightly, in a direction which should certainly cause no surprise. In § 3 we describe a special case of

Adams' counter-example, chosen to enable us to simplify the demonstration that it does indeed yield a counter-example to Statement 1. In fact this counter-example shows that actually there are spaces W, X, of the given kind such that $[W, X_Q] \to [W, X_0]$ fails to be injective for *any* non-empty Q! It is also noteworthy that, in Adams' counter-example, W and X are 1-connected.

A brief final section contains some remarks on the analogous situation when we consider free homotopy sets rather than based homotopy sets. An interesting question which then presents itself is the following. Let G be a finitely-generated nilpotent group and let G_0 be its rationalization. Let $\bar{G}$ be the set of conjugacy classes of G and define $\bar{G}_0$ similarly. Then $e\colon G \to G_0$ induces $e_*\colon \bar{G} \to \bar{G}_0$. Is e_* finite-to-one?

It should go without saying that we are very grateful to Professor Frank Adams for his well-founded scepticism and his perceptive suggestion of a counter-example.

2. The main results

We first consider Corollary II.5.4(a) of [**3**], which coincides with Corollary 2.2(a) of [**2**], and give a complete proof of it; this proof was only hinted at in our previous versions.

THEOREM 2.1. *Suppose W is a connected finite CW-complex and X is a nilpotent CW-complex of finite type. Let $S \subseteq T$ denote sets of primes. Then the canonical map $e_*\colon [W, X_T] \to [W, X_S]$ is finite-to-one.*

Proof. We may suppose that W^1 is a wedge of circles, and we first establish the conclusion when $W = W^1$. Then $e_*\colon [W, X_T] \to [W, X_S]$ is the localizing map $e\colon G_T \to G_S$, where G is a (finite) direct product of copies of $\pi_1 X$, and so a finitely-generated nilpotent group. Thus $\ker e$ is the S'-torsion of G_T, that is, the $(T \setminus S)$-torsion of G. But the torsion subgroup of G is finite, so that $\ker e$ is finite and the conclusion holds if $W = W^1$.

We may now proceed by induction. We assume that $W = V \cup e^n$ for $n \geqslant 2$, and that $e_*\colon [V, X_T] \to [V, X_S]$ is finite-to-one. Let $g\colon W \to X_T$ and let $\bar{g} = g \mid V$. We then have a commutative diagram (compare the proof of Theorem II.5.3 of [**3**])

$$(2.1)\qquad \begin{array}{ccccccc}
\pi_1(X_T^V, \bar{g}) & \xrightarrow{\varphi} & \pi_n X_T & \longrightarrow & [W, X_T]_g & \xrightarrow{\rho} & [V, X_T]_{\bar{g}} \\
\Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e_*} & & \Big\downarrow{\scriptstyle e_*} \\
\pi_1(X_S^V, e\bar{g}) & \xrightarrow{\psi} & \pi_n X_S & \longrightarrow & [W, X_S]_{eg} & \xrightarrow{\sigma} & [V, X_S]_{e\bar{g}}
\end{array}$$

Moreover, the horizontal sequences are exact in the sense that (in the notation of the top sequence) $\operatorname{coker}\varphi$ operates faithfully on $[W, X_T]_g$ and $\rho h = \rho g\ (= \bar{g})$ if and only if $h = g^\alpha$, where $\alpha \in \operatorname{coker}\varphi$. Now (2.1) induces a localizing map

$$e\colon \operatorname{coker}\varphi \to \operatorname{coker}\psi. \tag{2.2}$$

We now prove a crucial lemma which will also be used in our later results.

LEMMA 2.2. *The kernel of* $e\colon \operatorname{coker}\varphi \to \operatorname{coker}\psi$ *is finite.*

Proof. We have

$$\begin{tikzcd}
\pi_1(X_T^V, \bar{g}) \arrow[r, "\varphi"] \arrow[d, "e"] & \pi_n X_T \arrow[r, two heads] \arrow[d, "e"] & \operatorname{coker}\varphi \arrow[d, "e"] \\
\pi_1(X_S^V, e\bar{g}) \arrow[r, "\psi"] & \pi_n X_S \arrow[r, two heads] & \operatorname{coker}\psi
\end{tikzcd}$$

Now by Theorem II.3.11 of [**3**] $\pi_1(X_T^V, \bar{g})$ is T-local. Thus $\operatorname{coker}\varphi$ is T-local. Moreover, $\pi_n X_T$ is a finitely-generated $\mathbf{Z}_T$-module, and so therefore is $\operatorname{coker}\varphi$. It follows that $\operatorname{coker}\varphi = A_T \oplus B_T$, where A_T is a free $\mathbf{Z}_T$-module and B_T is a finite $\mathbf{Z}_T$-module. Then the S-localizing map

$$e\colon \operatorname{coker}\varphi \to \operatorname{coker}\psi$$

embeds A_T in A_S, say, and the kernel of e is just the S'-torsion of B_T and is hence finite.

We now return to the proof of Theorem 2.1. We prove that

(i) there exist finitely many $h\colon W \to X_T$ with given ρh and $e_* h$,

(ii) there exist finitely many ρh with given $e_* h$.

Obviously these two assertions together guarantee that there exist finitely many $h\colon W \to X_T$ with $eh = eg$, so that $e_*\colon [W, X_T] \to [W, X_S]$ is finite-to-one and the inductive step is complete. To prove (i) observe that if $\rho h = \bar{g}$ and $e_* h = eg$, then $h = g^\alpha$ with α in the kernel of (2.2), so that, by Lemma 2.2, α belongs to a finite set. To prove (ii) observe that if $e_* h = eg$ then $e_* \rho h = e_* \bar{g}$, so that ρh belongs to a finite set, by the inductive hypothesis. This completes the proof of Theorem 2.1.

We note the following obvious corollary.

COROLLARY 2.3. *The conclusion of Theorem* 2.1 *holds if* W *is quasifinite* (*nilpotent*).

For then there exists a map $h\colon \overline{W} \to W$ of a finite connected CW-complex into W inducing a bijection $h^*\colon [W, Y] \to [\overline{W}, Y]$ for any nilpotent space Y.

As we have said in the introduction, Corollary II.5.4(b) and (c) of [**3**], coinciding with Corollary 2.2(b) and (c) of [**2**], are false, in general. We may however replace them by the following weaker statement; here we say that a function of pointed sets is *weakly injective* if the counter-image of the base point is the base point.

THEOREM 2.4. *Suppose W is a connected finite CW-complex and X is a nilpotent CW-complex of finite type. Then there exists a cofinite set of primes Q such that the canonical map $e_*: [W, X_S] \to [W, X_0]$ is weakly injective for all $S \subseteq Q$.*

We postpone the proof since the theorem is an immediate consequence of Theorem 2.10 below.

COROLLARY 2.5. *The conclusion of Theorem 2.4 holds if W is quasifinite (nilpotent).*

COROLLARY 2.6. *If, in addition to the hypotheses of Theorem 2.4, W is a co-loop (for example, a suspension or a 1-connected co-H-space), then there exists a cofinite set of primes Q such that the canonical map*

$$e_*: [W, X_S] \to [W, X_0]$$

is injective for all $S \subseteq Q$.

For if W is a co-loop, then e_* is injective if it is weakly injective.

COROLLARY 2.7. *If, in addition to the hypotheses of Theorem 2.4, X is a rational H-space, then there exists a cofinite set of primes Q such that the canonical map $e_*: [W, X_S] \to [W, X_0]$ is injective for all $S \subseteq Q$.*

Proof. Suppose $\dim W \leqslant n$. We may kill the homotopy groups of X above dimension n to produce Y and it is then plain that Y is a nilpotent CW-complex of finite type, and a rational H-space, and that the map $X \to Y$ induces bijections $[W, X_P] \to [W, Y_P]$ for all sets of primes P. Thus we may suppose that X itself has vanishing homotopy groups in dimensions not less than $n+1$.

Now consider the map $X \times X \to X_0 \times X_0 \to X_0$, where the first map is rationalization and the second is the H-structure on X_0. If

$$u: (X \times X)^{n+1} \to X_0$$

is the restriction of this map, then by Theorem 2.10 below, we may find a cofinite set of primes Q_1 such that both u and $u \mid (X \vee X)^{n+1}$ lift uniquely to X_{Q_1}. Let the lift of u be $v_1: (X \times X)^{n+1} \to X_{Q_1}$. Since the homotopy groups of X_{Q_1} vanish in dimension greater than n, v_1 extends uniquely to

$v_2 \colon X \times X \to X_{Q_1}$, inducing $v \colon X_{Q_1} \times X_{Q_1} \to X_{Q_1}$. It is now plain that v is an H-structure on X_{Q_1} which makes $e \colon X_{Q_1} \to X_0$ an H-map.

Thus we may endow $[W, X_S]$, for all $S \subseteq Q_1$, with a loop-structure such that $e_* \colon [W, X_S] \to [W, X_0]$ is a homomorphism of loops. If

$$e_* \colon [W, X_S] \to [W, X_0]$$

is weakly injective for all $S \subseteq Q_2$, where Q_2 is cofinite, and if $Q = Q_1 \cap Q_2$, then Q is cofinite and $e_* \colon [W, X_S] \to [W, X_0]$ is a weakly injective homomorphism for all $S \subseteq Q$. But a weakly injective homomorphism of loops is injective.

REMARKS. (i) Again we may suppose W is quasifinite instead of finite in Corollary 2.7.

(ii) We could improve the 'duality' between Corollaries 2.6 and 2.7 by merely requiring, in the former, that W have the rational homology type of a (finite) co-loop.

We now turn our attention to Corollary II.5.11 of [**3**], which coincides with Corollary 3.5 of [**2**] and Theorem 2 of [**1**]. This result is correct but we must give a new proof since our original argument was based on the strong form of Corollary II.5.4(c) of [**3**]. By way of preparation we strengthen the statements of two purely algebraic results in [**3**]; it is plain that a very mild adaptation of the proofs given there serves to establish the stronger statements.

LEMMA 2.8. (cf. Lemma I.3.4 of [**3**]). *If $G \in \mathrm{N}$ is finitely generated, then there exists a cofinite set of primes P such that $G_S \to G_0$ is injective for all $S \subseteq P$.*

THEOREM 2.9. (cf. Theorem I.3.5 of [**3**]). *Let G, $K \in \mathrm{N}$ be finitely generated and let $\varphi \colon G \to K_0$. Then there exists a cofinite set of primes Q such that φ has a unique lift into K_S for all $S \subseteq Q$.*

We are now ready to state and prove a strengthened form of Corollary II.5.11 of [**3**].

THEOREM 2.10. *Suppose W is a connected finite CW-complex and X is a nilpotent CW-complex of finite type. Given a map $f \colon W \to X_0$, there exists a cofinite set of primes Q such that f has a unique lift (up to homotopy) into X_S for all $S \subseteq Q$.*

Proof. We argue as in Theorem 2.1. If $W = W^1$, a wedge of circles, the conclusion follows from Theorem 2.9. Thus we may assume that $W = V \cup e^n$ for $n \geqslant 2$, and that there exists a cofinite set of primes $\bar{Q}$ such that $\bar{f} = f \mid V$ has a unique lift into X_T for all $T \subseteq \bar{Q}$. By Corollary II.5.10 of [3] there exists a cofinite set of primes R such that f lifts into X_R. Let $Q^* = \bar{Q} \cap R$. Then Q^* is cofinite, and if f lifts to $g^*\colon W \to X_{Q^*}$ with $\bar{g}^* = g^* \mid V$, we have the diagram (compare (2.1))

$$\begin{array}{ccccc} \pi_1(X_{Q^*}^V, \bar{g}^*) & \xrightarrow{\varphi^*} & \pi_n(X_{Q^*}) & \longrightarrow & [W, X_{Q^*}]_{g^*} \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e_*} \\ \pi_1(X_0^V, \bar{f}) & \xrightarrow{\psi} & \pi_n(X_0) & \longrightarrow & [W, X_0]_f \end{array} \tag{2.3}$$

Now $e\colon \operatorname{coker}\varphi^* \to \operatorname{coker}\psi$ has finite kernel (Lemma 2.2). Let P be the (finite) set of primes involved in this kernel and let $Q = Q^* \cap P'$. Then Q is cofinite. Let $S \subseteq Q$ and let $g\colon W \to X_S$ arise by dropping g^*; set $\bar{g} = g \mid V$. Then (2.3) may be expanded to

$$\begin{array}{ccccc} \pi_1(X_{Q^*}^V, \bar{g}^*) & \xrightarrow{\varphi^*} & \pi_n(X_{Q^*}) & \longrightarrow & [W, X_{Q^*}]_{g^*} \\ \Big\downarrow{\scriptstyle e_1} & & \Big\downarrow{\scriptstyle e_1} & & \Big\downarrow{\scriptstyle e_{1*}} \\ \pi_1(X_S^V, \bar{g}) & \xrightarrow{\varphi} & \pi_n(X_S) & \longrightarrow & [W, X_S]_g \\ \Big\downarrow{\scriptstyle e_2} & & \Big\downarrow{\scriptstyle e_2} & & \Big\downarrow{\scriptstyle e_{2*}} \\ \pi_1(X_0^V, \bar{f}) & \xrightarrow{\psi} & \pi_n(X_0) & \longrightarrow & [W, X_0]_f \end{array} \tag{2.4}$$

From (2.4) we obtain $\operatorname{coker}\varphi^* \xrightarrow{e_1} \operatorname{coker}\varphi \xrightarrow{e_2} \operatorname{coker}\psi$, where e_1 localizes at S and e_2 rationalizes. Now all the torsion of $\operatorname{coker}\varphi^*$ belongs to the set P and $S \subseteq P'$. Thus $\operatorname{coker}\varphi$ is torsion-free, so that

$$e_2\colon \operatorname{coker}\varphi \to \operatorname{coker}\psi$$

is injective. We now extract from (2.4) to construct the diagram

$$\begin{array}{ccccc} \operatorname{coker}\varphi & \longrightarrow & [W, X_S]_g & \longrightarrow & [V, X_S]_{\bar{g}} \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e_*} & & \Big\downarrow{\scriptstyle e_*} \\ \operatorname{coker}\psi & \longrightarrow & [W, X_0]_f & \longrightarrow & [V, X_0]_{\bar{f}} \end{array} \tag{2.5}$$

from which we will infer that g is the unique lift of f into X_S. For let h be another lift and let $\bar{h} = h \mid V$. Then $e_*(\bar{h}) = e_*(\bar{g}) = \bar{f}$; but $S \subseteq Q^* \subseteq \bar{Q}$, so that, by the inductive hypothesis, $\bar{h} = \bar{g}$. It follows that $h = g^{\alpha}$ for a unique $\alpha \in \operatorname{coker}\varphi$. Then, applying e_*, we have $f = f^{e\alpha}$. It follows that $e\alpha$ is the neutral element of $\operatorname{coker}\psi$, so that α is the neutral element of $\operatorname{coker}\varphi$ and $h = g$, as required. This achieves the inductive step and completes the proof of the theorem.

Corollary 2.11. *The conclusion of Theorem* 2.10 *holds if W is quasi-finite* (*nilpotent*).

Remark. Theorem 2.10 is stronger than Corollary II.5.11 of [**3**], or, equivalently, Corollary 3.5 of [**2**], since it asserts that a lift of $f: W \to X_0$ to X_S is unique for all $S \subseteq Q$. In principle it would appear possible that f could lift uniquely to X_Q, but there might be several lifts to some X_S with $S \subseteq Q$. However, this is *not* possible in the category **N**, and we conjecture that it is not possible in **NH**. In **N** we have

Proposition 2.12. *Let $G, K \in$ N and let $\varphi: G \to K_0$ lift to K_Q. Then every lift of φ to K_S, with $S \subseteq Q$, lifts further to K_Q.*

Proof. Let $K_Q \xrightarrow{e_1} K_S \xrightarrow{e_2} K_0$ be localizing maps. Suppose we are given $\varphi: G \to K_0$, $\psi, \psi': G \to K_S$, $\theta: G \to K_Q$, with $e_1\theta = \psi$, $e_2\psi = e_2\psi' = \varphi$. If $x \in G$ then $\psi'x = \psi x.u_x$, where $u_x \in \ker e_2$. Now $\ker e_2$ is the S-torsion subgroup of K_S, which may be identified with the S-torsion subgroups of K and K_Q. In other words, we may regard e_1 as being the identity on S-torsion subgroups and it thus makes sense to define a function $\theta': G \to K_Q$ by $\theta'x = \theta x.u_x$. Obviously $e_1\theta' = \psi'$, so it remains to show that θ' is a homomorphism. Now

$$\theta'x.\theta'y = \theta x.u_x.\theta y.u_y,$$

while $\theta'xy = \theta xy.u_{xy}$. Thus, since θ is a homomorphism it remains to show that $u_x.\theta y.u_y = \theta y.u_{xy}$, or

$$\theta y^{-1}.u_x.\theta y.u_y = u_{xy}. \tag{2.6}$$

Since ψ' is a homomorphism it follows that the two sides of (2.6) are the same under e_1. But each of the two sides lies in the S-torsion subgroup, so (2.6) is established and with it Proposition 2.12.

Note that there is no gain in generality in writing K_Q instead of K; we merely wished to bring the notation into line with the enunciations of this section.

Corollary 2.13. *Let $G, K \in$ N and let $\varphi: G \to K_0$. If φ lifts uniquely to K_Q it lifts uniquely to every K_S, for $S \subseteq Q$.*

3. The Adams counter-example

In this section we show that Corollary II.5.4(c) of [3] is false in general.

To simplify the argument we specialize Adams' example as follows. Let m be a positive integer and let us designate by $m: S^2 \vee S^3 \to S^3 \vee S^3$ the map which is constant on S^2 and maps S^3 to the second target S^3 with degree m. Let $\omega: S^4 \to S^2 \vee S^3$ be the Whitehead product map $[\iota_1, \iota_2]$, where ι_1, ι_2 embed S^2, S^3 in $S^2 \vee S^3$. There is then an exact sequence

$$(3.1) \qquad \pi_1(S^3 \vee S^{3^V}, m) \xrightarrow{\varphi} \pi_5(S^3 \vee S^3) \longrightarrow [S^2 \times S^3, S^3 \vee S^3],$$

where $V = S^2 \vee S^3$; here φ itself, and, in particular, its image, depend very strongly on the choice of m, as we shall see. Note that

$$\pi_5(S^3 \vee S^3) = \mathbf{Z} \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2,$$

the $\mathbf{Z}$-summand being generated by the Whitehead product w, of i_1, i_2, where i_1, i_2 embed each S^3 in $S^3 \vee S^3$. Consider the element of

$$\pi_1(S^3 \vee S^{3^V}, m)$$

represented by $f: (S^2 \vee S^3) \times I \to S^3 \vee S^3$, given by

$$f(x,t) = \rho(x,t), o \quad (x \in S^2),$$
$$f(y,t) = o, my \qquad (y \in S^3),$$

where $\rho: S^2 \times I \to S^3$ is the evident identification map of degree 1 and my stands for the image of y under a map $S^3 \to S^3$ of degree m. It is easy to see that $\varphi f = mw$.

We will show that for any element ξ of $\pi_1(S^3 \vee S^{3^V}, m)$, the image of 2ξ under φ is a multiple of mw. It then follows that, if m is odd,

$$\operatorname{coker} \varphi = \mathbf{Z}/m \oplus T,$$

where T is a finite 2-group. Now $\operatorname{coker}\varphi$ embeds in $[S^2 \times S^3, S^3 \vee S^3]$. For any m, $\operatorname{coker}\varphi$ is annihilated by rationalization. However, given any cofinite Q, we choose m to belong to Q and then the generator of $\mathbf{Z}/m$ in $\operatorname{coker}\varphi$ is not annihilated by Q-localization. Thus for no cofinite Q is $e_*: [S^2 \times S^3, S^3_Q \vee S^3_Q] \to [S^2 \times S^3, S^3_0 \vee S^3_0]$ injective. Notice that, as pointed out in the introduction, we have here an example where the primes entering into $|\ker(\operatorname{coker}\varphi_g \to \operatorname{coker}\psi_{eg})|$ depend on g, and are unbounded in number.

Thus it remains to establish our claim that $\varphi(2\xi)$ is a multiple of mw. Let us write $\psi: I^n, \dot{I}^n \to S^n, o$ for the canonical map (homeomorphic on the interior of I^n to $S^n \setminus o$). Then ω is represented by the map $\dot{I}^5 \to S^2 \vee S^3$ given by

$$(a,b) \mapsto (\psi a, o) \quad (a \in I^2,\ b \in \dot{I}^3); \quad (a,b) \mapsto (o, \psi b) \quad (a \in \dot{I}^2,\ b \in I^3).$$

A map $(S^2 \vee S^3) \times I \to S^3 \vee S^3$, representing ξ, is a pair of maps

$$u\colon S^2 \times I \to S^3 \vee S^3$$

mapping $S^2 \times \dot{I}$ to o, and $v\colon S^3 \times I \to S^3 \vee S^3$ mapping $S^3 \times \dot{I}$ by m. Hence $\varphi\xi$ is represented by a map $\dot{I}^5 \times I \to S^3 \vee S^3$, given by

$$(a,b,t) \mapsto u(\psi a, t) \quad (a \in I^2,\ b \in \dot{I}^3); \quad (a,b,t) \mapsto v(\psi b, t) \quad (a \in \dot{I}^2,\ b \in I^3).$$

We extend this to a map of $(I^5 \times I)^{\cdot}$ to $S^3 \vee S^3$ by setting the image of (a,b,t) equal to $m\psi b$ for $a \in I^2$, $b \in I^3$, $t \in \dot{I}$; and the resulting map again represents $\varphi\xi$.

It is plain that this map $\dot{I}^6 \to S^3 \vee S^3$ represents a Whitehead product of some element of $\pi_3(S^3 \vee S^3)$ with mi_2, provided that $v(\psi b, t)$ is (up to homotopy) independent of t. Now two candidates for v (given the restriction of v to $S^3 \times \dot{I}$) differ by an element of $\pi_4(S^3 \vee S^3) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Thus if we double the original element ξ we achieve our objective, as claimed.†

4. Free homotopy classes

In this final section we show, by a much simpler counter-example than that given in § 3, that the analogue of Corollary II.5.4(c) of [3] is also false when we consider free homotopy classes instead of based homotopy classes. We also briefly discuss the analogues of Corollaries 2.6 and 2.7 of this paper in the case of free homotopy; we show that the analogue of the former is false while the analogue of the latter is true.

Let us write $[W, X]_{\mathrm{fr}}$ for the set of free homotopy classes of maps from W to X; here W will be connected finite and X will be nilpotent, of finite type. We will show that there exist W and X such that

$$e_*\colon [W, X_Q]_{\mathrm{fr}} \to [W, X_0]_{\mathrm{fr}}$$

is injective for no cofinite set of primes Q; indeed it will fail to be injective whenever Q is non-empty.

Thus we take $W = S^1$ and $X = K(G, 1)$, with $G = U(3, \mathbf{Z})$, the set of 3×3 matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad (a, b, c \in \mathbf{Z}).$$

Then $[W, X]_{\mathrm{fr}}$ may, of course, be identified with the set of conjugacy classes of elements of G, whatever the group G may be. We now prove, for our particular choice of G,

† It is clear that we have really shown that $e_*\colon [S^2 \times S^3, S^3_Q \vee S^3_Q] \to [S^2 \times S^3, S^3_0 \vee S^3_0]$ is not injective for *any* non-empty Q.

LEMMA 4.1. *G is nilpotent of class 2 and, for any set of primes P, $G_P = U(3, \mathbf{Z}_P)$, the set of 3×3 matrices*

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad (a, b, c \in \mathbf{Z}_P),$$

with the obvious localization $e: G \to G_P$.

Proof. We may identify G with the set of triples (a, b, c), where $a, b, c \in \mathbf{Z}$, together with the product rule

$$(4.1) \qquad (a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2).$$

It is plain from (4.1) that the centre $Z(G)$ of G consists of triples $(0, 0, c)$, and that $G/Z(G) \cong \mathbf{Z} \oplus \mathbf{Z}$. Thus we have a central extension

$$\mathbf{Z} \rightarrowtail G \twoheadrightarrow \mathbf{Z} \oplus \mathbf{Z}.$$

Similarly $U(3, \mathbf{Z}_P)$ may be represented as a central extension

$$\mathbf{Z}_P \rightarrowtail U(3, \mathbf{Z}_P) \twoheadrightarrow \mathbf{Z}_P \oplus \mathbf{Z}_P$$

and, moreover, the embedding $e: \mathbf{Z} \rightarrowtail \mathbf{Z}_P$ extends to a map of central extensions

$$(4.2) \qquad \begin{tikzcd} \mathbf{Z} \arrow[r, tail] \arrow[d, "e"] & G \arrow[r, two heads] \arrow[d] & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, "e"] \\ \mathbf{Z}_P \arrow[r, tail] & U(3, \mathbf{Z}_P) \arrow[r, two heads] & \mathbf{Z}_P \oplus \mathbf{Z}_P \end{tikzcd}$$

From (4.2) we see that G is nilpotent of class 2 and that the natural map $G \to U(3, \mathbf{Z}_P)$ P-localizes, as claimed.

Thus to establish the claim made in our second paragraph we must show that

$$\overline{U(3, \mathbf{Z}_Q)} \to \overline{U(3, \mathbf{Q})}$$

is not injective for any non-empty Q, where $\bar{H}$ is the set of conjugacy classes of the group H. To this end, consider the elements $A = (0, n, 0)$, $B = (0, n, m)$, where $m, n \in \mathbf{Z} \backslash \{0\}$, of the group $U(3, \mathbf{Z})$; here we use the simplified notation of the proof of the lemma. A straightforward calculation shows that $T \in U(3, \mathbf{Q})$ satisfies $TAT^{-1} = B$ if and only if $T = (m/n, b, c)$, with arbitrary $b, c \in \mathbf{Q}$. Thus, given any non-empty set of primes Q we have only to choose n to be a Q-number not equal to 1 and $m = 1$ in order to find two elements A, B which are conjugate in $U(3, \mathbf{Q})$ and not in $U(3, \mathbf{Z}_Q)$.

REMARKS. 1. This example is of particular interest because, of course, $U(3, \mathbf{Z}_T) \to U(3, \mathbf{Z}_S)$ is injective for *all* pairs $S \subseteq T$.

2. This same example shows that the analogue of Corollary 2.6 is false in free homotopy, since S^1 is a suspension. On the other hand, the analogue of Corollary 2.7 does hold:

PROPOSITION 4.2. *Let W be a connected finite CW-complex and let X be a nilpotent CW-complex of finite type which is a rational H-space. Then there exists a cofinite set of primes Q such that the canonical map*

$$e_* : [W, X_S]_{\mathrm{fr}} \to [W, X_0]_{\mathrm{fr}}$$

is injective for all $S \subseteq Q$.

Proof. We know that there exists Q such that $e_* : [W, X_S] \to [W, X_0]$ is injective for all $S \subseteq Q$. But, since X_0 is an H-space it follows that $[W, X_0] \to [W, X_0]_{\mathrm{fr}}$ is bijective. Thus we have a commutative diagram, with $S \subseteq Q$,

$$\begin{tikzcd}
{[W, X_S]} \arrow[r, two heads] \arrow[d, tail, "e_*"] & {[W, X_S]_{\mathrm{fr}}} \arrow[d, "e_*"] \\
{[W, X_0]} \arrow[r, tail, two heads] & {[W, X_0]_{\mathrm{fr}}}
\end{tikzcd}$$

showing that $e_* : [W, X_S]_{\mathrm{fr}} \to [W, X_0]_{\mathrm{fr}}$ is injective.

REMARK. Notice that, by choosing the same Q as in Corollary 2.7, we have ensured that $[W, X_S] \to [W, X_S]_{\mathrm{fr}}$ is, in fact, bijective for $S \subseteq Q$, even without insisting that X_S be an H-space. Of course, in the *proof* of Corollary 2.7, Q was chosen in such a way that X_S had a canonical H-space structure.

COROLLARY 4.3. *Let W be a finite CW-complex and let X be a nilpotent CW-complex of finite type which is a rational H-space. Then there exists a cofinite set of primes Q such that the canonical maps $e_* : [W, X_S] \to [W, X_0]$, $e_* : [W, X_S]_{\mathrm{fr}} \to [W, X_0]_{\mathrm{fr}}$ are injective for all $S \subseteq Q$.*

Proof. We simply argue for each component of W, invoking Corollary 2.7 or Proposition 4.2.

REFERENCES

1. P. HILTON, G. MISLIN, and J. ROITBERG, 'Topological localization and nilpotent groups', *Bull. Amer. Math. Soc.* 78 (1972) 1060–63.

2. P. HILTON, G. MISLIN, and J. ROITBERG, 'Homotopical localization', *Proc. London Math. Soc.* 26 (1973) 693–706.
3. —— —— —— *Localization of nilpotent groups and spaces*, Mathematics Studies 15 (North Holland, Amsterdam, 1975).

PETER HILTON
Battelle Seattle Research Center
and
Case Western Reserve University
Cleveland, Ohio 44106

GUIDO MISLIN
Eidgenössische Technische Hochschule
Zürich, Switzerland

JOSEPH ROITBERG
Hunter College and Graduate Center of CUNY
New York, NY 10021